

ON THE HOMOLOGY OF THE DUAL DE RHAM COMPLEX

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ABSTRACT. We study the homology of the dual de Rham complex as functors on the category of abelian groups. We give a description of homology of the dual de Rham complex up to degree 7 for free abelian groups and present a corrected version of the proof of Jean's computations of the zeroth homology group.

1.1. Divided power functor. Let \mathbf{Ab} be the category of abelian groups. Recall the definition of the graded divided power functor (see [9]) $\Gamma_* = \bigoplus_{n \geq 0} \Gamma_n : \mathbf{Ab} \rightarrow \mathbf{Ab}$. The graded abelian group $\Gamma_*(A)$ is generated by symbols $\gamma_i(x)$ of degree $i \geq 0$ satisfying the following relations for all $x, y \in A$:

- 1) $\gamma_0(x) = 1$
- 2) $\gamma_1(x) = x$
- 3) $\gamma_s(x)\gamma_t(x) = \binom{s+t}{s} \gamma_{s+t}(x)$
- 4) $\gamma_n(x+y) = \sum_{s+t=n} \gamma_s(x)\gamma_t(y), \quad n \geq 1$
- 5) $\gamma_n(-x) = (-1)^n \gamma_n(x), \quad n \geq 1.$

In particular, the canonical map $A \simeq \Gamma_1(A)$ is an isomorphism. The following additional properties of elements of the abelian group $\Gamma(A)$ will be useful ($x, y \in A, r \geq 1$):

$$\begin{aligned} \gamma_r(nx) &= n^r \gamma_r(x), \quad n \in \mathbb{Z}; \\ r\gamma_r(x) &= x\gamma_{r-1}(x); \\ x^r &= r! \gamma_r(x); \\ \gamma_r(x)y^r &= x^r \gamma_r(y). \end{aligned}$$

A direct computation implies that

$$\Gamma_r(\mathbb{Z}/n) \simeq \mathbb{Z}/n(r, n^\infty),$$

where (r, n^∞) is the limit $\lim_{m \rightarrow \infty} (r, n^m)$. The degree 2 component $\Gamma_2(A)$ of the divided power algebra is the Whitehead functor $\Gamma(A)$. It is the universal group for homogenous quadratic maps from A into abelian groups.

1.2. Dual de Rham complex. Let A be an abelian group. For $n \geq 1$, denote by SP^n and Λ^n the n th symmetric and exterior power functors respectively. For $n \geq 1$, let $D_*^n(A)$ and $C_*^n(A)$ be the complexes of abelian groups defined by

$$\begin{aligned} D_i^n(A) &= SP^i(A) \otimes \Lambda^{n-i}(A), \quad 0 \leq i \leq n, \\ C_i^n(A) &= \Lambda^i(A) \otimes \Gamma_{n-i}(A), \quad 0 \leq i \leq n, \end{aligned}$$

where the differentials $d^i : D_i^n(A) \rightarrow D_{i-1}^n(A)$ and $d_i : C_i^n(A) \rightarrow C_{i-1}^n(A)$ are:

$$d^i((b_1 \dots b_i) \otimes b_{i+1} \wedge \dots \wedge b_n) = \sum_{k=1}^i (b_1 \dots \hat{b}_k \dots b_i) \otimes b_k \wedge b_{i+1} \wedge \dots \wedge b_n$$

$$d_i(b_1 \wedge \dots \wedge b_i \otimes X) = \sum_{k=1}^i (-1)^k b_1 \wedge \dots \wedge \hat{b}_k \wedge \dots \wedge b_i \otimes b_k X$$

for any $X \in \Gamma_{n-i}(A)$. The complex $D^n(A)$ is the degree n component of the classical de Rham complex, first introduced in the present context of polynomial functors in [4] and denoted Ω_n in [5]. The dual complexes $C^n(A)$ were considered in [6]. We will call them the dual de Rham complexes.

The dual de Rham complexes appear naturally in the theory of homology of Eilenberg-Mac Lane spaces. Let A be a free abelian group. There are well-known natural isomorphisms (see, for example, [1]):

$$\begin{aligned} H_n K(A, 1) &= \Lambda^n(A), \quad n \geq 1 \\ H_{2n} K(A, 2) &= \Gamma_n(A), \quad n \geq 1 \\ H_{2n+1} K(A, 2) &= 0, \quad n \geq 0. \end{aligned}$$

Consider the path-fibration:

$$K(A, 2) \rightarrow PK(A, 3) \rightarrow K(A, 3)$$

and the homology spectral sequence

$$E_{p,q}^2 = H_p K(A, 3) \otimes H_q K(A, 2) \Rightarrow \mathbb{Z}[0]$$

The dual de Rham complexes can be recognized as natural parts of the E^3 -term of this spectral sequence. For example, we have the following natural diagrams:

$$\begin{array}{ccccccc} E_{6,0}^3 & \xrightarrow{d_{6,0}^3} & E_{3,2}^3 & \xrightarrow{d_{3,2}^3} & E_{0,4}^3 & & \\ \parallel & & \parallel & & \parallel & & \\ \Lambda^2(A) & \xrightarrow{d_2} & A \otimes A & \xrightarrow{d_1} & \Gamma_2(A), & & \end{array}$$

$$\begin{array}{ccccccc} E_{9,0}^3 & \xrightarrow{d_{9,0}^3} & E_{6,2}^3 & \xrightarrow{d_{6,2}^3} & E_{3,5}^3 & \xrightarrow{d_{3,4}^3} & E_{0,6}^3 \\ \uparrow & & \parallel & & \parallel & & \parallel \\ \Lambda^3(A) & \xrightarrow{d_3} & \Lambda^2(A) \otimes A & \xrightarrow{d_2} & A \otimes \Gamma_2(A) & \xrightarrow{d_1} & \Gamma_3(A) \end{array}$$

We will now give a functorial description of certain homology groups of these complexes $C^n(A)$. Some applications of these results in the theory of derived functors one can find in [2].

Proposition 1.1. *Let A be a free abelian. Then*

- (1) [5] *For any prime number p , $H_0 C^p(A) = A \otimes \mathbb{Z}/p$, and $H_i C^p(A) = 0$, for all $i > 0$;*

(2) [6] *There is a natural isomorphism*

$$H_0 C^n(A) \simeq \bigoplus_{p|n, p \text{ prime}} \Gamma_{n/p}(A \otimes \mathbb{Z}/p).$$

We will make use of the following fact from number theory (see [7] corollary 2):

Lemma 1.1. *Let n and k be a pair positive integers and p is a prime number, then*

$$\binom{pn}{pk} \equiv \binom{n}{k} \pmod{p^r},$$

where r is the largest power of p dividing $pnk(n-k)$.

Proof of Proposition 1.1 (2). Let $n \geq 2$ and define the map

$$q_n : \Gamma_n(A) \rightarrow \bigoplus_{p|n} \Gamma_{n/p}(A \otimes \mathbb{Z}/p)$$

by setting:

$$q_n : \gamma_{i_1}(a_1) \dots \gamma_{i_t}(a_t) \mapsto \sum_{p|i_k, \text{ for all } 1 \leq k \leq t} \gamma_{i_1/p}(\bar{a}_1) \dots \gamma_{i_t/p}(\bar{a}_t),$$

$$i_1 + \dots + i_t = n, a_k \in A, \bar{a}_k \in A \otimes \mathbb{Z}/p.$$

If $(i_1, \dots, i_t) = 1$, then we set

$$q_n(\gamma_{i_1}(a_1) \dots \gamma_{i_t}(a_t)) = 0, \text{ where } a_k \in A \text{ for all } k.$$

Let us check that the map q_n is well-defined. For that we have to show that

$$q_n(\gamma_{j_1}(x)\gamma_{j_2}(x) \dots \gamma_{j_t}(x_t)) = q_n\left(\binom{j_1 + j_2}{j_1} \gamma_{j_1+j_2}(x) \dots \gamma_{j_t}(x_t)\right) \quad (1.1)$$

$$q_n(\gamma_{j_1}(x_1 + y_1) \dots \gamma_{j_t}(x_t)) = \sum_{k+l=j_1} q_n(\gamma_k(x_1)\gamma_l(y_1) \dots \gamma_{j_t}(x_t)) \quad (1.2)$$

$$q_n(\gamma_{j_1}(-x_1) \dots \gamma_{j_t}(x_t)) = (-1)^{j_1} q_n(\gamma_{j_1}(x_1) \dots \gamma_{j_t}(x_t)) \quad (1.3)$$

Verification of (1.1). First suppose that $p|j_1 + j_2$, $p \nmid j_1$. Since for every pair of numbers $n \geq k$, one has

$$\frac{n}{(n, k)} \mid \binom{n}{k}, \quad (1.4)$$

we have

$$\binom{j_1 + j_2}{j_1} = p^s m_1, \quad j_1 + j_2 = p^s m_2, \quad (m_1, p) = (m_2, p) = 1, \quad s \geq 1.$$

Hence $\binom{j_1 + j_2}{j_1} \frac{m_2}{m_1} = p(\frac{j_1 + j_2}{p})$. Observe that $\Gamma_{\frac{j_1 + j_2}{p}}(A \otimes \mathbb{Z}/p)$ is a p -group, hence

$$\binom{j_1 + j_2}{j_1} \gamma_{\frac{j_1 + j_2}{p}}(\bar{x}) = 0, \quad \bar{x} \in A \otimes \mathbb{Z}/p \quad (1.5)$$

since

$$\binom{j_1 + j_2}{j_1} \frac{m_2}{m_1} \gamma_{\frac{j_1 + j_2}{p}}(\bar{x}) = p \bar{x} \gamma_{\frac{j_1 + j_2}{p} - 1}(\bar{x}) = 0.$$

The equality (1.5) implies that

$$\begin{aligned} & q_n\left(\binom{j_1+j_2}{j_1}\gamma_{j_1+j_2}(x)\dots\gamma_{j_t}(x_t)\right) - q_n(\gamma_{j_1}(x)\gamma_{j_2}(x)\dots\gamma_{j_t}(x_t)) = \\ & \binom{j_1+j_2}{j_1} \sum_{p|j_1+j_2, p|j_k, k>2} \gamma_{\frac{j_1+j_2}{p}}(\bar{x})\dots\gamma_{j_t/p}(\bar{x}_t) - \sum_{p|j_k, \text{ for all } 1\leq k\leq t} \gamma_{j_1/p}(\bar{x})\gamma_{j_2/p}(\bar{x})\dots\gamma_{j_t/p}(\bar{x}_t) = \\ & \left(\binom{j_1+j_2}{j_1} - \binom{\frac{j_1+j_2}{p}}{\frac{j_1}{p}}\right) \sum_{p|j_k, \text{ for all } 1\leq k\leq t} \gamma_{\frac{j_1+j_2}{p}}(\bar{x})\dots\gamma_{j_t/p}(\bar{x}_t) \end{aligned}$$

Let $j_1 + j_2 = p^s m$, $(m, p) = 1$. Lemma 1.1 implies that

$$\binom{j_1+j_2}{j_1} \equiv \binom{\frac{j_1+j_2}{p}}{\frac{j_1}{p}} \pmod{p^r}$$

where r is the largest power of p , dividing $(j_1 + j_2)j_1j_2/p^2$. Since $p|j_1$, $p|j_2$, we have

$$\binom{j_1+j_2}{j_1} \equiv \binom{\frac{j_1+j_2}{p}}{\frac{j_1}{p}} \pmod{p^s}.$$

Hence

$$\left(\binom{j_1+j_2}{j_1} - \binom{\frac{j_1+j_2}{p}}{\frac{j_1}{p}}\right) \gamma_{\frac{j_1+j_2}{p}}(\bar{x}) = 0, \quad \bar{x} \in A \otimes \mathbb{Z}/p$$

and the property (1.1) follows.

Verification of (1.2). We have

$$\begin{aligned} & q_n(\gamma_{j_1}(x_1 + y_1)\dots\gamma_{j_t}(x_t)) - \sum_{k+l=j_1} q_n(\gamma_k(x_1)\gamma_l(y_1)\dots\gamma_{j_t}(x_t)) = \\ & \sum_{p|j_t, t\geq 1} \gamma_{j_1/p}(\bar{x}_1 + \bar{y}_1)\dots\gamma_{j_t/p}(\bar{x}_t) - \sum_{k+l=j_1} \sum_{p|k, p|l, p|j_t, t\geq 2} \gamma_{k/p}(\bar{x}_1)\gamma_{l/p}(\bar{y}_1)\dots\gamma_{j_t/p}(\bar{x}_t) = \\ & \sum_{p|j_t, t\geq 1} \gamma_{j_1/p}(\bar{x}_1 + \bar{y}_1)\dots\gamma_{j_t/p}(\bar{x}_t) - \sum_{p|k, p|l, p|j_t, t\geq 2} \sum_{\frac{k}{p} + \frac{l}{p} = \frac{j_1}{p}} \gamma_{k/p}(\bar{x}_1)\gamma_{l/p}(\bar{y}_1)\dots\gamma_{j_t/p}(\bar{x}_t) = 0 \end{aligned}$$

Verification of (1.3). We have

$$\begin{aligned} & q_n(\gamma_{j_1}(-x_1)\dots\gamma_{j_t}(x_t)) - (-1)^{j_1} q_n(\gamma_{j_1}(x_1)\dots\gamma_{j_t}(x_t)) = \\ & \sum_{p|j_k, k\geq 1} (\gamma_{j_1/p}(-\bar{x}_1) - (-1)^{j_1} \gamma_{j_1/p}(\bar{x}_1))\dots\gamma_{j_t/p}(\bar{x}_t) = 0 \end{aligned}$$

since

$$(\gamma_{j_1/p}(-\bar{x}_1) - (-1)^{j_1} \gamma_{j_1/p}(\bar{x}_1)) = 0$$

(we separately check the cases $p = 2$ and $p \neq 2$).

We now know that the map q_n is well-defined. It induces a map

$$\bar{q}_n : H^0 C^n(A) \rightarrow \bigoplus_{p|n} \Gamma_{n/p}(A \otimes \mathbb{Z}/p),$$

since $q_n(a) = 0$ for every $a \in \text{im}\{A \otimes \Gamma_{n-1}(A) \rightarrow \Gamma_n(A)\}$.

Let $A = \mathbb{Z}$, then

$$H_0(\mathbb{Z}) = \text{coker}\{\Gamma_{n-1}(\mathbb{Z}) \otimes \mathbb{Z} \rightarrow \Gamma_n(\mathbb{Z})\} \simeq \text{coker}\{\mathbb{Z} \xrightarrow{n} \mathbb{Z}\} \simeq \mathbb{Z}/n.$$

Let $n = \prod p_i^{s_i}$ be the prime decomposition of n . Then

$$\bigoplus_{p_i|n} \Gamma_{n/p_i}(\mathbb{Z}/p_i) = \bigoplus_{p_i|n} \mathbb{Z}/p_i^{s_i} = \mathbb{Z}/n.$$

It follows from definition of the map \bar{q}_n , that

$$\bar{q}_n : H_0 C^n(\mathbb{Z}) \rightarrow \bigoplus_{p_i|n} \Gamma_{n/p_i}(\mathbb{Z}/p_i)$$

is an isomorphism.

For free abelian groups A and B , one has a natural isomorphism of complexes

$$C^m(A \oplus B) \simeq \bigoplus_{i+j=n, i,j \geq 0} C^i(A) \otimes C^j(B)$$

This implies that the cross-effect¹ $C^n(A|B)$ of the functor $C^n(A)$ is described by

$$C^m(A|B) \simeq \bigoplus_{i+j=n, i,j > 0} C^i(A) \otimes C^j(B)$$

and its homology $(H_k C^n)(A|B) = H_k C^n(A|B)$ can be described with the help of Künneth formulas:

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n, i,j > 0, r+s=k} H_r C^i(A) \otimes H_s C^j(B) &\rightarrow H_k C^n(A|B) \rightarrow \\ &\bigoplus_{i+j=n, i,j > 0, r+s=k-1} \text{Tor}(H_r C^i(A), H_s C^j(B)) \rightarrow 0 \end{aligned}$$

Hence we have the following simple description of the cross-effect of $H_0 C^n(A)$:

$$H_0 C^n(A|B) \simeq \bigoplus_{i+j=n, i,j > 0} H_0 C^i(A) \otimes H_0 C^j(B) \quad (1.6)$$

From the other hand, we have the following decomposition of the cross-effect of the functor $\tilde{\Gamma}_{n/p}^p(A) := \Gamma_{n/p}(A \otimes \mathbb{Z}/p)$:

$$\tilde{\Gamma}_{n/p}^p(A|B) = \bigoplus_{l+k=n/p} \Gamma_l(A \otimes \mathbb{Z}/p) \otimes \Gamma_k(B \otimes \mathbb{Z}/p)$$

Hence

$$\bigoplus_{p|n} \tilde{\Gamma}_{n/p}^p(A|B) = \bigoplus_{p|n} \bigoplus_{l+k=n/p} \Gamma_l(A \otimes \mathbb{Z}/p) \otimes \Gamma_k(B \otimes \mathbb{Z}/p) \quad (1.7)$$

We must now show that the maps \bar{q}_n preserve the decompositions (1.6) and (1.7). This is equivalent to the commutativity of the following diagram:

¹Given a functor $F : \mathbf{Ab} \rightarrow \mathbf{Ab}$, its cross effect is defined as the kernel of the natural map $F(A|B) = \ker\{F(A \oplus B) \rightarrow F(A) \oplus F(B)\}$, $A, B \in \mathbf{Ab}$

$$\begin{array}{ccc}
H_0 C^i(A) \otimes H_0 C^j(B) & \xrightarrow{\bar{q}_i \otimes \bar{q}_j} & \bigoplus_{p|i} \Gamma_{i/p}(A \otimes \mathbb{Z}/p) \otimes \bigoplus_{p|j} \Gamma_{j/p}(B \otimes \mathbb{Z}/p) \\
\downarrow \wr & & \downarrow \varepsilon' \\
H_0(C^i(A) \otimes C^j(B)) & & \\
\downarrow & & \\
H_0 C^{i+j}(A \oplus B) & \xrightarrow{\bar{q}_{i+j}} & \bigoplus_{p|(i+j)} \Gamma_{\frac{i+j}{p}}((A \oplus B) \otimes \mathbb{Z}/p)
\end{array} \quad (1.8)$$

The map $\varepsilon' \circ (\bar{q}_i \otimes \bar{q}_j)$ is defined via the natural map

$$\prod \gamma_{i_k}(x_k) \otimes \prod \gamma_{j_k}(y_k) \mapsto \sum_{p|i_k, p|j_k} \prod \gamma_{i_k/p}(\bar{x}_i) \prod \gamma_{j_k/p}(\bar{y}_i) \in \bigoplus_{p|(i+j)} \Gamma_{\frac{i+j}{p}}((A \oplus B) \otimes \mathbb{Z}/p),$$

$$x_k \in A, y_k \in B, \bar{x}_k \in A \otimes \mathbb{Z}/p, \bar{y}_k \in B \otimes \mathbb{Z}/p, \sum j_k = j, \sum i_k = i$$

and the commutativity of the diagram (1.8) follows. This proves that the natural map

$$H_0 C^n(A|B) \simeq \bigoplus_{i+j=n, i,j>0} H_0 C^i(A) \otimes H_0 C^j(B) \rightarrow \bigoplus_{p|n} \tilde{\Gamma}_{n/p}^p(A|B) = \bigoplus_{p|n} \bigoplus_{l+k=n/p} \Gamma_l(A \otimes \mathbb{Z}/p) \otimes \Gamma_k(B \otimes \mathbb{Z}/p)$$

induced by \bar{q}_n on cross-effects is an isomorphism, and it follows from this that \bar{q}_n is an isomorphism for all free abelian groups A . \square

1.3. Derived functors and homology. Let A be an abelian group, and F an endofunctor on the category of abelian groups. Recall that for every $n \geq 0$ the derived functor of F in the sense of Dold-Puppe [3] are defined by

$$L_i F(A, n) = \pi_i(F K P_*[n]), \quad i \geq 0$$

where $P_* \rightarrow A$ is a projective resolution of A , and K is the Dold-Kan transform, inverse to the Moore normalization functor

$$N : \text{Simpl}(\mathbf{Ab}) \rightarrow C(\mathbf{Ab})$$

from simplicial abelian groups to chain complexes.

Recall the description of the highest derived functors of the tensor power functor due to Mac Lane [8]. The group $\text{Tor}^{[n]}(A)$ is generated by the n -linear expressions $\tau_h(a_1, \dots, a_n)$ (where all a_i live in the subgroup ${}_h A$ of elements a of A for which $ha = 0$ ($h > 0$)), subject to the so-called slide relations

$$\tau_{hk}(a_1, \dots, a_i, \dots, a_n) = \tau_h(ka_1, \dots, ka_{i-1}, a_i, ka_{i+1}, \dots, ka_n) \quad (1.9)$$

for all i whenever $hka_j = 0$ for all $j \neq i$ and $ha_i = 0$. The associativity of the derived tensor product functor implies that there are canonical isomorphisms

$$\text{Tor}^{[n]}(A) \simeq \text{Tor}(\text{Tor}^{[n-1]}(A), A), \quad n \geq 2.$$

For $n \geq 2$, there is a natural isomorphism:

$$L_{n-1} \otimes^n (A) \simeq \text{Tor}^{[n]}(A).$$

The map $\otimes^n \longrightarrow SP^n$ induces a natural epimorphism

$$\mathrm{Tor}^{[n]}(A) \rightarrow L_{n-1}SP^n(A) \quad (1.10)$$

which sends the generators $\tau_h(a_1, \dots, a_n)$ of $\mathrm{Tor}^{[n]}(A)$ to generators $\beta_h(a_1, \dots, a_n)$ of

$$\mathcal{S}_n(A) := L_{n-1}SP^n(A).$$

The kernel of this map is generated by the elements $\tau_h(a_1, \dots, a_n)$ with $a_i = a_j$ for some $i \neq j$. It is shown by Jean in [6] that

$$L_i SP^n(A) \simeq (L_i SP^{i+1}(A) \otimes SP^{n-(i+1)}(A)) / \mathrm{Jac}_{SP}, \quad (1.11)$$

where Jac_{SP} is the subgroup generated by elements of the form

$$\sum_{k=1}^{i+2} (-1)^k \beta_h(x_1, \dots, \hat{x}_k, \dots, x_{i+2}) \otimes x_k y_1 \dots y_{n-i-2}.$$

with $x_i \in {}_h A$ and $y_j \in A$ for all i, j .

We will now construct a series of maps:

$$f_i^{n,p} : L_i SP^{n/p}(A \otimes \mathbb{Z}/p) \rightarrow H_i C^n(A)$$

for a free abelian A and $p|n$. We first choose liftings (x_i) to A of a given family of elements $(\bar{x}_i) \in A \otimes \mathbb{Z}_p$. We set

$$\begin{aligned} f_i^{n,p} : \beta_p(\bar{x}_1, \dots, \bar{x}_{i+1}) \otimes \bar{x}_{i+2} \dots \bar{x}_{\frac{n}{p}} &\mapsto \eta_i(x_1, \dots, x_{\frac{n}{p}}) := \\ \sum_{t=1}^{i+1} (-1)^t x_1 \wedge \dots \wedge \hat{x}_t \wedge \dots \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \dots \widehat{\gamma_{p-1}(x_t)} \dots \gamma_{p-1}(x_{i+1}) &\gamma_p(x_t) \gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}}) \\ &\in \Lambda^i(A) \otimes \Gamma_{n-i}(A), \quad \bar{x}_k \in A \otimes \mathbb{Z}/p, \quad x_k \in A. \end{aligned}$$

Proposition 1.2. *The maps $f_i^{n,p}$ are well defined for all i, n, p .*

Proof. We have

$$\begin{aligned} \eta_i(px_1, \dots, x_{\frac{n}{p}}) &= \\ \sum_{t=1}^{i+1} (-1)^t p x_1 \wedge \dots \wedge \hat{x}_t \wedge \dots \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \dots \widehat{\gamma_{p-1}(x_t)} \dots \gamma_{p-1}(x_{i+1}) &\gamma_p(x_t) \gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}}) \\ \sum_{t=1}^{i+1} (-1)^t x_1 \wedge \dots \wedge \hat{x}_t \wedge \dots \wedge x_{i+1} \otimes x_t \gamma_{p-1}(x_1) \dots \gamma_{p-1}(x_t) \dots \gamma_{p-1}(x_{i+1}) &\gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}}) = \\ d_{i+1}(x_1 \wedge \dots \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \dots \gamma_{p-1}(x_t) \dots \gamma_{p-1}(x_{i+1}) &\gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}})) \\ &\in \mathrm{im}\{\Lambda^{i+1}(A) \otimes \Gamma_{n-i-1}(A) \xrightarrow{d_{i+1}} \Lambda^i(A) \otimes \Gamma_{n-i}(A)\} \end{aligned}$$

One verifies that for every $1 \leq k \leq n/p$, one has

$$\eta_i(x_1, \dots, px_k, \dots, x_{\frac{n}{p}}) \in \mathrm{im}\{\Lambda^{i+1}(A) \otimes \Gamma_{n-i-1}(A) \xrightarrow{d_{i+1}} \Lambda^i(A) \otimes \Gamma_{n-i}(A)\}$$

It follows that the map $f_i^{n,p} : L_i SP^{n/p}(A \otimes \mathbb{Z}/p) \rightarrow (\Lambda^i(A) \otimes \Gamma_{n-i}(A))/\text{im}(d_{i+1})$ is well-defined. The simplest examples of such elements are the following

$$\begin{aligned}\eta_1(x_1, x_2) &= x_1 \otimes x_1 \gamma_2(x_2) - x_2 \otimes x_2 \gamma_2(x_1) \in A \otimes \Gamma_2(A), \\ \eta_2(x_1, x_2, x_3) &= -x_1 \wedge x_2 \otimes x_1 x_2 \gamma_2(x_3) + x_1 \wedge x_3 \otimes x_1 x_3 \gamma_2(x_2) \\ &\quad - x_2 \wedge x_3 \otimes x_2 x_3 \gamma_2(x_1) \in \Lambda^2(A) \otimes \Gamma_4(A)\end{aligned}$$

By construction, the elements η_i lie in $\Lambda^i(A) \otimes \Gamma_{n-i}(A)$. In fact, let us verify that

$$\eta_i(x_1, \dots, x_{\frac{n}{p}}) \in \ker\{\Lambda^i(A) \otimes \Gamma_{n-i}(A) \xrightarrow{d_i} \Lambda^{i-1}(A) \otimes \Gamma_{n-i+1}(A)\}$$

Observe that

$$\begin{aligned}d_i \eta_i(x_1, \dots, x_{\frac{n}{p}}) &= \\ d_i \sum_{t=1}^{i+1} (-1)^t x_1 \wedge \dots \wedge \hat{x}_t \wedge \dots \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \dots \widehat{\gamma_{p-1}(x_t)} \dots \gamma_{p-1}(x_{i+1}) \gamma_p(x_t) \gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}})\end{aligned}$$

In this sum, for every pair of indexes $1 \leq r < s \leq i+1$ there occurs a pair of terms

$$(-1)^{s+r} x_1 \wedge \dots \wedge \hat{x}_r \dots \wedge \hat{x}_s \dots \wedge x_{i+1} \otimes x_r \gamma_{p-1}(x_1) \dots \widehat{\gamma_{p-1}(x_s)} \dots \gamma_{p-1}(x_{i+1}) \gamma_p(x_r) \gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}})$$

and

$$(-1)^{s+r+1} x_1 \wedge \dots \wedge \hat{x}_r \dots \wedge \hat{x}_s \dots \wedge x_{i+1} \otimes x_s \gamma_{p-1}(x_1) \dots \widehat{\gamma_{p-1}(x_r)} \dots \gamma_{p-1}(x_{i+1}) \gamma_p(x_r) \gamma_p(x_{i+2}) \dots \gamma_p(x_{\frac{n}{p}})$$

which cancel each other. It follows that the entire sum is equal to zero.

For the same reason, the map

$$\begin{aligned}\sum_{k=1}^{i+2} (-1)^k \beta_p(x_1, \dots, \hat{x}_k, \dots, x_{i+2}) \otimes x_k y_1 \dots y_l \mapsto \\ \sum_{k=1}^{i+2} (-1)^k \sum_{m=1, m < k}^{i+2} (-1)^m x_1 \wedge \dots \wedge \hat{x}_m \dots \wedge \hat{x}_k \dots \wedge x_{i+2} \otimes \\ \left(\prod_{l=1, l \neq m, k}^{i+2} \gamma_{p-1}(x_l) \right) \gamma_p(x_m) \gamma_p(x_k) \gamma_p(y_1) \dots \gamma_p(y_l) + \\ \sum_{k=1}^{i+2} (-1)^k \sum_{m=1, m > k}^{i+2} (-1)^{m+1} x_1 \wedge \dots \wedge \hat{x}_k \dots \wedge \hat{x}_m \dots \wedge x_{i+2} \otimes \\ \left(\prod_{l=1, l \neq m, k}^{i+2} \gamma_{p-1}(x_l) \right) \gamma_p(x_m) \gamma_p(x_k) \gamma_p(y_1) \dots \gamma_p(y_l)\end{aligned}$$

is trivial. □

Given abelian group A and $i > 0$, consider a natural map

$$f_i^n = \sum_{p|n} f_i^{n,p} : \bigoplus_{p|n, p \text{ prime}} L_i SP_p^n(A \otimes \mathbb{Z}/p) \rightarrow H_i C^n(A)$$

Theorem 1.1. *The map f_i^n is an isomorphism for $i > 0$, $n \leq 7$.*

Proof. Given an abelian simplicial group $(G_\bullet, \partial_i, s_i)$, let $A(G_\bullet)$ be the associated chain complex with $A(G_\bullet)_n = G_n$, $d_n = \sum_{i=0}^n (-1)^i \partial_i$. Recall that given abelian simplicial groups G_\bullet and H_\bullet , the Eilenberg-Mac Lane map

$$g : A(G_\bullet) \otimes A(H_\bullet) \rightarrow A(G_\bullet \otimes H_\bullet)$$

is given by

$$g(a_p \otimes b_q) = \sum_{(p; q)\text{-shuffles } (\mu, \nu)} (-1)^{\text{sign}(\mu, \nu)} s_{\nu_q} \dots s_{\nu_1}(a_p) \otimes s_{\mu_p} \dots s_{\mu_1}(b_q)$$

For any free abelian group A and infinite cyclic group B , we will show that there is a natural commutative diagram with vertical Künneth short exact sequences:

$$\begin{array}{ccc} \bigoplus_{i+j=\frac{n}{p}, r+s=k} L_r SP^i(A \otimes \mathbb{Z}/p) \otimes L_s SP^j(B \otimes \mathbb{Z}/p) & \longrightarrow & \bigoplus_{i+j=n, r+s=k} H_r C^i(A) \otimes H_s C^j(B) \\ \downarrow & & \downarrow \\ L_k SP^{\frac{n}{p}}((A \oplus B) \otimes \mathbb{Z}/p) & \xrightarrow{f_k^{n,p}} & H_k C^n(A \oplus B) \\ \downarrow & & \downarrow \\ \bigoplus_{i+j=\frac{n}{p}, r+s=k-1} \text{Tor}(L_r SP^i(A), L_s SP^j(B)) & \longrightarrow & \bigoplus_{i+j=n, r+s=k-1} \text{Tor}(H_r C^i(A), H_s C^j(B)) \end{array} \quad (1.12)$$

where all maps are induced by maps $f_*^{*,p}$. Since B is a cyclic, it is enough to consider the case $s = 0$ and summands of the upper square from (1.12)

$$\begin{array}{ccc} L_r SP^i(A \otimes \mathbb{Z}/p) \otimes SP^j(B \otimes \mathbb{Z}/p) & \xrightarrow{f_r^{ip,p} \otimes f_0^{jp,p}} & H_r C^{ip}(A \otimes \mathbb{Z}/p) \otimes H_0 C^{jp}(B \otimes \mathbb{Z}/p) \\ \downarrow s_r & & \downarrow h_r \\ L_r SP^{i+j}((A \oplus B) \otimes \mathbb{Z}/p) & \xrightarrow{f_r^{ip+jp,p}} & H_r C^{ip+jp}((A \oplus B) \otimes \mathbb{Z}/p), \end{array} \quad (1.13)$$

where the maps s_r, h_r come from Künneth exact sequences. Consider natural projections

$$u_1 : L_r SP^{r+1}(A \otimes \mathbb{Z}/p) \otimes SP^{i-r-1}(A \otimes \mathbb{Z}/p) \rightarrow L_r SP^i(A \otimes \mathbb{Z}/p)$$

$$u_2 : L_r SP^{r+1}((A \oplus B) \otimes \mathbb{Z}/p) \otimes SP^{i+j-r-1}((A \oplus B) \otimes \mathbb{Z}/p) \rightarrow L_r SP^{i+j}((A \oplus B) \otimes \mathbb{Z}/p)$$

The map s_r is defined by

$$s_r : u_1(\beta_p(a_1, \dots, a_{r+1}) \otimes a_{r+2} \dots a_i) \otimes b_1 \dots b_j \mapsto u_2(\beta_p(a_1, \dots, a_{r+2}) \otimes a_{r+2} \dots a_i b_1 \dots b_j),$$

$a_k \in A \otimes \mathbb{Z}/p, b_l \in B \otimes \mathbb{Z}/p$

We have

$$f_r^{ip,p} \otimes f_0^{jp,p}(u_1(\beta_p(a_1, \dots, a_{r+1}) \otimes a_{r+2} \dots a_i) \otimes b_1 \dots b_j) = \eta_r(a_1, \dots, a_i) \otimes \gamma_p(b_1) \dots \gamma_p(b_j)$$

and we see that the diagram (1.13) is commutative.

The map f_i^n is an isomorphism for a cyclic group A , since both source and target groups are trivial. For $i = 1, n = 4$, and cyclic B , we have a natural diagram

$$\begin{array}{ccc} L_1 SP^2(A \otimes \mathbb{Z}/2 | B \otimes \mathbb{Z}/2) & \longrightarrow & H_1 C^4(A | B) \\ \uparrow \wr & & \uparrow \wr \\ \text{Tor}(A \otimes \mathbb{Z}/2, B \otimes \mathbb{Z}/2) & \longrightarrow & \text{Tor}(H_0 C^2(A), H_0 C^2(B)) \end{array}$$

and the isomorphism f_1^4 follows. The proof is similar for other i, n . The only non-trivial case here is $i = 1, n = 6$, for the 2-torsion component of $H_1 C^6(A)$. In that case, the statement follows from the natural isomorphism

$$\text{Tor}(SP^2(A \otimes \mathbb{Z}/2), \mathbb{Z}/2) \rightarrow \text{Tor}(\Gamma_2(A \otimes \mathbb{Z}/2), \mathbb{Z}/2)$$

for every free abelian group A . □

Remark 1.1. For any free abelian group A and prime number p , there are canonical isomorphisms

$$\begin{aligned} L_{n-1} SP^n(A \otimes \mathbb{Z}/p) &\simeq \Lambda^n(A \otimes \mathbb{Z}/p) \\ L_i SP^n(A \otimes \mathbb{Z}/p) &= \text{coker}\{\Lambda^{i+2}(A \otimes \mathbb{Z}/p) \otimes SP^{n-i-2}(A \otimes \mathbb{Z}/p) \xrightarrow{\kappa_{i+2}} \\ &\quad \Lambda^{i+1}(A \otimes \mathbb{Z}/p) \otimes SP^{n-i-1}(A \otimes \mathbb{Z}/p)\} \end{aligned}$$

where κ_i is the corresponding differential in the n -th Koszul complex.

When $i = 1, n = 3$ and A a free abelian group there is a natural isomorphism

$$L_1 SP^3(A \otimes \mathbb{Z}/p) \simeq \mathcal{L}^3(A \otimes \mathbb{Z}/p),$$

where \mathcal{L}^3 is the third Lie functor (see [2], for example). Observe however that the natural map

$$f_1^8 : L_1 SP^4(A \otimes \mathbb{Z}/2) \rightarrow H_1 C^8(A)$$

is not an isomorphism. Indeed, every element of $L_1 SP^4(A \otimes \mathbb{Z}/2)$ is 2-torsion, whereas $H_1 C^8(A)$ can contain 4-torsion elements, since its cross-effect $H_1 C^8(A | B)$ contains $\text{Tor}(\Gamma_2(A \otimes \mathbb{Z}/2), \Gamma_2(B \otimes \mathbb{Z}/2))$ as a subgroup. The map f_1^8 is given by

$$\beta_2(\bar{a}, \bar{b}) \otimes \bar{c}\bar{d} \mapsto a \otimes a\gamma_2(b)\gamma_2(c)\gamma_2(d) - b \otimes b\gamma_2(a)\gamma_2(c)\gamma_2(d), \quad a, b, c, d \in A.$$

The following table, which is a consequence theorem 1.1, gives a complete description of $H_i C^n(A)$ for $n \leq 7$ and A free abelian:

q	$H_0 C^q(A)$	$H_1 C^q(A)$	$H_2 C^q(A)$	$H_3 C^q(A)$
7	$A \otimes \mathbb{Z}/7$	0	0	0
6	$\Gamma_2(A \otimes \mathbb{Z}/3) \oplus \Gamma_3(A \otimes \mathbb{Z}/2)$	$\Lambda^2(A \otimes \mathbb{Z}/3) \oplus \mathcal{L}^3(A \otimes \mathbb{Z}/2)$	$\Lambda^3(A \otimes \mathbb{Z}/2)$	0
5	$A \otimes \mathbb{Z}/5$	0	0	0
4	$\Gamma_2(A \otimes \mathbb{Z}/2)$	$\Lambda^2(A \otimes \mathbb{Z}/2)$	0	0
3	$A \otimes \mathbb{Z}/3$	0	0	0
2	$A \otimes \mathbb{Z}/2$	0	0	0

For example, the isomorphism

$$f : \Lambda^2(A \otimes \mathbb{Z}/2) \rightarrow H_1 C^4(A) \tag{1.14}$$

is defined, for representatives $a, b \in A$ of $\bar{a}, \bar{b} \in A \otimes \mathbb{Z}/2$, by

$$f : \bar{a} \otimes \bar{b} \mapsto a \otimes a\gamma_2(b) - b \otimes b\gamma_2(a).$$

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